Introduction To Calculus
Limits

In collaboration with
Josue Arreaga
Teacher Assistant

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Abstract

Let's begin our introduction to Calculus by stating that it is by no means abstract mathematics or a course required to graduate. Calculus is very real and it comes with a great variety of applications. After this course, students will acquire the abilities to model real life scenarios such as cars moving through an interval of time, graphing their velocity, graphing their acceleration, and finding the car's displacement through an integral. Furthermore, students will acquire knowledge on how to maximize profits for a business, how to minimize costs of production, and optimize as a whole. For now however, we will begin our introduction with the building block of calculus known as limits. A limit allows us to view graphs closely (way too close) and are the foundation for derivatives which in turn lead to anti-derivatives (integrals).

1 Limits

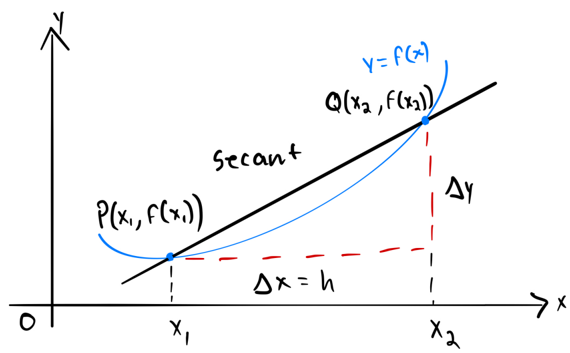
In this chapter we develop the concept of a limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only a small change in $f(x)$. Other functions can have values jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish among these behaviors.

1.1 Definition 1 - Average Rate of Change

We begin this chapter by looking into the Average Rate of Change definition (ARC). This principle can be broken down into saying rise over run. Basically we need to know that if we plug in an x_1 value and we then plug in a x_2 value we will have different y outputs regardless as to how small the difference in x may be. In this section of limits precision matter a lot. In other words $0.999 \neq 0.9999 \neq 0.99999 \neq 1$.

This definition states that the average rate of change $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\delta y}{\delta x} = \frac{f(x_1) - f(x_2)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \text{h} \neq 0$$



In this case notice that h is Δx in other words h represents the change from x_1 to x_2 .

1.2 Theorem 1 - Limit laws

Limit laws theorem provide us with a few algebraic rules that allow us to simplify otherwise difficult problems into simple ones. It is therefore recommended to learn them, similar to basic algebra we multiply, divide, add, substract and so on.

This theorem states that if *L,M,c, and k* are real numbers and

$$\lim_{x\to c} f(x) = \text{L and } \lim_{x\to c} g(x) = \text{K, then}$$

1. *Sum Rule:* $\lim_{x\to a} (f(x) + g(x)) = L + K$
2. *Difference Rule:* $\lim_{x\to a} (f(x) - g(x)) = L - K$
3. *Constant Multiple Rule:* $\lim_{x\to a} (k \times f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to a} (f(x) \times g(x)) = L \cdot K$
5. *Quotient Rule:* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, $K \neq 0$
6. *Power Rule:* $\lim_{x\to a} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule* $\lim_{x\to a} \sqrt[n]{f(x)} = L^{1/n}$, n a positive integer

1.3 Sum Rule Proof

We will now go ahead and prove the Sum Rule, (this is by no means simple, and we do not expect the students to prove this at such an early level). If you would like to give it a shot however, try to prove the others.

Proof of Sum Rule: *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = K + L$

First let $\epsilon > 0$ then because $\lim_{x \to c} f(x) = \text{K}$ and $\lim_{x \to c} g(x) = \text{L}$ there is a $\delta_1 > 0$ and a $\delta_2 > 0$ such that,

$$|f(x) - \text{K}| < \frac{e}{2} \text{ whenever } 0 < |x - a| < \delta_1$$

$$|f(x) - \text{L}| < \frac{e}{2} \text{ whenever } 0 < |x - a| < \delta_2$$

Now choose $\delta = \min \delta_1, \delta_2$. Then we need to show that

$$|f(x) + g(x) - (\text{K+L})| < e \text{ whenever } 0 < |x - a| < \delta$$

Assume that $0 < |x - a| < \delta$. We then have,

$$|f(x) + g(x) - (\text{K+L})| = |(f(x) - \text{K}) + (g(x) - \text{L})|$$

$$\leq |(f(x) - \text{K}) + (g(x) - \text{L})| \text{ by the triangle inequality}$$

$$< \frac{e}{2} + \frac{e}{2}$$

$$= e$$

In the third step we used the fact that, by our choice of δ, we also have $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ and so we can use the initial statements in our proof.

Try to prove the difference law *hint: it's very similar*

1.4 Theorem 2 - Limit of Polynomials

The main thing we need to know about Theorem 2 is that when we have the function of a polynomial and we are approaching a value denoted by c we can take its limit as x approaches the value c. Basically when $\lim_{x \to c}$ we can substitute x with c just like in a regular function $f(x)$ we can have substitute value x for c.

The theorem formally states:

$$\text{If } P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0 \text{ , then}$$
$$\lim_{x \to c} P(x) = \textit{Sum Rule: } \lim_{x \to c} \ (f(x) + g(x)) \ = L + K$$

Proof In order to prove this theorem we will use induction of the degree of $p(x)$. To begin induction we first prove that the result is true for the polynomials of degree 0 and degree 1. In this process we will use the following two limit law rules:

$$1)\textit{Product Rule: } \lim_{x \to c}(f(x) \times g(x)) = L \cdot K$$
$$4)\textit{Sum Rule: } \lim_{x \to c} \ (f(x) + g(x)) \ = L + K$$

Let us first start with the polynomials of degree 0. In this case $p(x)$ is just a constant k. We have thus $p(x)$ = k for all x and hence in particular $p(a) = k$. Now we need to show that $\lim_{x \to a} p(x) = p(a)$ or $\lim_{x \to a} k = k$. This is simple enough as we see that variable x is not involved and thus limit of k remains k.

This is a simple as $|f(x)$ - L$|$ = $|k$ -$k|$ = 0 which is always less than e for whatever δ we choose.

Next we handle polynomials of degree 1. Let $p(x) = Ax + B$. Then we have $p(a)$ = Aa + B. We have

$$\lim_{x \to a} p(x) = \lim_{x \to a} \ \mathrm{A}a + \mathrm{B}$$
$$= \lim_{x \to a} \mathrm{A} \cdot \lim_{x \to a} x + \lim_{x \to a} \mathrm{B} \text{ (using limit rules (1) and (2))}$$
$$= \mathrm{A} \lim_{x \to a} x + \mathrm{B}$$
$$= \mathrm{A}a + \mathrm{B} = p(a)$$

The last line uses the simple result that $\lim_{x \to a} x = a$ Recall that a formal proof requires that for every $e > 0$ we find a $\delta > 0$ such that $|f(x)$ - L$| < e$ whenever $0 < |x$ - $a| < \delta$ where $f(x) = x$ and L = a.

In this case $|f(x)$ - L$|$ = $|x$ - $a|$ so that $\delta = e$ will do the job.

Now we assume that the result holds for any polynomial $p(x)$ of degrees n so that if $p(x)$ is any polynomial of degree n then $\lim_{x \to a} p(x) = p(a)$. Let us now try to see what we get it $p(x)$ is of degree $(n + 1)$. Clearly if $p(x)$ is of degree $(n + 1)$ we can write

$$p(x) = a_0 x^{n+1} + a_1 x^n + ... + a_n x + a_{n+1}$$
$$= x(\ a_0 x^n + a_1 x^{n-1} + ... + a_n) + a_{n+1}$$
$$= x \cdot q(x) + k$$

where $q(x)$ is a polynomial of degree n and k is some constant k = a_{n+1}. We then see that $p(a) = a \cdot q(a) + k$. We can now proceed in the following manner

$$\lim_{x \to a} = \{x \cdot q(x) + k\}$$

$$= \lim_{x \to a} q(x) \cdot \lim_{x \to a} \ q(x) + \lim_{x \to a} k \text{ (using rules (1) and (4))}$$

$$a \cdot q(a) + k \text{ (result holds for polynomial } q(x) \text{ of degree } n)$$

$$= p(a)$$

The proof by induction is now complete.

1.5 Theorem 3 - Limits of rational functions

I would like for the reader to notice that this theorem is **rule number 3** of limit laws. Why is this a theorem on its own? Perhaps the best answer is that in calculus the idea of asymptotes, horizontal and vertical is extremely useful.

This theorem states that If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

Proof In order to prove this theorem we will start by showing that

$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{L}$ Let $e > 0$. We'll not need this right away, but these proofs always start off with this statement.

Now, Because $\lim_{x \to a} g(x) = \text{L}$ there is a $\delta_1 > 0$ such that,

$$|g(x) \text{ - L}| < \frac{|L|}{2} \text{ whenever } 0 < |x \text{ - } a| < \delta_1$$

Now, assuming that $0 < |x \text{ - } a| < \delta_1$ we have,

$$|\text{L}| = |\text{L - } g(x) + g(x)| \text{ , just adding zero to L}$$

$$\leq |\text{L - } g(x)| + |g(x)| \text{ , using the triangle inequality}$$

$$\leq |g(x) \text{ - L}| + |g(x)| \text{ , recall that } |\text{L - } g(x)| = |g(x) \text{ - L}|$$

$$< \frac{|L|}{2} + |g(x)| \text{ , assuming that } 0 < |x \text{ - } a| < \delta_1$$

Rearranging this gives,

$$|\text{L}| < \frac{|L|}{2} + |g(x)| \longrightarrow \frac{|L|}{2} < |g(x)| \longrightarrow \frac{1}{|g(x)|} < \frac{2}{|L|}$$

Now, there is also a $\delta_2 > 0$ such that,

$$|g(x \text{ - L})| < \frac{|L|^2}{2} e \text{ whenever } 0 < |x \text{ - } a| < \delta_2$$

choose $\delta = \min \delta_1, \delta_2$. if $0 < |x \text{ - } a| < \delta$ we have,

$$|\frac{1}{g(x)} \text{ - } \frac{1}{L}| = |\frac{L - g(x)}{Lg(x)}| \text{ , common denominators}$$

$$= \frac{1}{|Lg(x)|} |\text{L - } g(x)| \text{ , doing a little rewriting}$$

$$= \frac{1}{|L|} \frac{1}{|g(x)} |g(x \text{ - L})| \text{ , doing more rewriting}$$

$$< \frac{1}{|L|} \tfrac{2}{|L|} |g(x) \text{ - L}| \text{ , assuming that } 0 < |x \text{ - } a| < \delta \leq \delta_1$$

$$< \frac{2}{|L|^2} \frac{|L|^2}{2} e$$

$$= e$$

Now that we've proven $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{L}$ the more general fact is easy.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} [f(x) \frac{1}{g(x)}]$$

$$= \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)}$$

$$= \text{K} \frac{1}{L} = \frac{K}{L}$$

This completes our proof.

1.6 Theorem 4 - The Squeeze Theorem

The Squeeze theorem works similar to a sandwich. Think of a Philly cheese stake sub. In the middle we have the steak, cheese, sauce, etc. The point is that the middle is bounded by the upper and lower slices of bread. If we know where the upper and lower slices are, we then know where the steak, cheese, sauce, etc is. We do a similar thing with unknown points of functions to find their limit. We grab an upper function, a middle(unknown) function and a lower function. Then we compute a given point this means we squeeze really hard and find out unknown value for the function at that point. Fun fact, the Squeeze theorem is also known as the Sandwich Theorem.

This theorem states that if the functions *f, g, and h*have the property that for all x in an open interval containing c, except possibly at c itself,

$$f(x) \leq g(x) \leq h(x) \text{ and if}$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = \text{L, then}$$

$$\lim_{x \to c} g(x) = \text{L}$$

Proof since $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = \text{L}$, then for any number $e > 0$, there are positive numbers δ_1 and δ_2 so that

$$\text{whenever } 0 < |x \text{ - } c| < \delta_1 \text{ then } |f(x) \text{ - L}| < e$$
$$\text{whenever } 0 < |x \text{ - } c| < \delta_2 \text{ then } |h(x) \text{ - L}| < e$$

Choose δ to be the smaller of the numbers δ_1 and δ_2. Then $0 <|x \text{ - } c| < \delta$ implies that both $|f(x) \text{ - L}| < e$ and $|h(x) \text{ - L}| < e$. In other words, $0 < |x \text{ - } c| < \delta$ implies that both

$$\text{L - } e < f(x) < \text{L} + e \text{ and L - } e < h(x) < \text{L} + e$$

Since $f(x) \leq g(x) \leq h(x)$ for all $x \neq c$ in the open interval, it follows that whenever $0 < |x \text{ - } c| < \delta$ and x is in the open interval, we have

$$\text{L - } e < f(x) \leq g(x) \leq h(x) < \text{L} + \epsilon$$

Then for any given number $e > 0$, there is a positive number δ so that whenever $0 < |x \text{ - } c| < \delta$, then L - $e < f(x) \leq g(x) \leq h(x) <$ L + e, or equivalently, $|g(x) \text{ - L}| < e$. That is, $\lim_{x \to c} g(x) = \text{L}$, This completes our proof.

1.7 Simplifying

Up to this point lets review what has occurred.

1. We stated that a small change in x will produce a small change in y. This is similar to the concept taught in algebra or earlier of rise over run.
2. We begun to show that the limits of functions $\lim_{x \to c} f(x)$ share properties that are identical to basic functions $f(x)$. For instance we can substitute x for a value c, we can add the result of functions and so on.
3. We used the idea of bounds to help us find an unknown value.

1.8 Definition 2 - Epsilon Delta

The Epsilon Delta definition of a limit although very wordy, states a concept similar to that of the average rate of change, (rise over run). It is also used a lot of the time to prove theorems. Here is its formal definition:

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \to c} f(x) = L,$$

If, for every number $e > 0$, there exists a corresponding number $\delta > 0$ such that

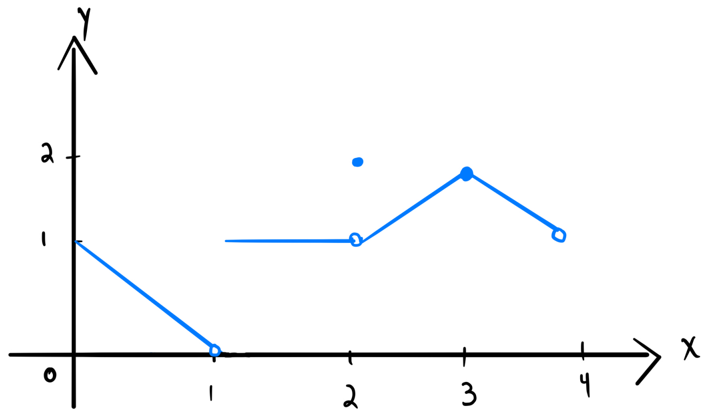
$$|f(x) - L\ | < e \text{ whenever } 0 < |x - c| < \delta$$

1.9 Theorem 5

Suppose that a function f is defined on an open interval containing c, except in perhaps at c itself. Then *f(x)* has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one sided limits are equal:

$$\lim_{x\to c} f(x) = L \leftrightarrow \lim_{x\to c^-} f(x) = L \leftrightarrow \lim_{x\to c^+} f(x) = L$$

Exercise In order to visualize and understand this theorem better lets look at the following graph and find the limits of the points $x = 0, 1, 2, 3, 4$.



At $x = 0$: $\lim_{x\to 0^-} f(x) =$ does not exist , $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1} f(x) = 1$

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$, $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1} f(x) =$ does not exists

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$ and $\lim_{x\to 2^+} f(x) = 1$ therefore, $\lim_{x\to 1} f(x) = 1$

At $x = 3$: $\lim_{x\to 3^-} f(x) = 0$, $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1} f(x) =$ does not exists

At $x = 4$ $\lim_{x\to 4^-} f(x) = 1$, $\lim_{x\to 4^+} f(x)=$ does not exist , $\lim_{x\to 1} f(x) = 1$

1.10 Definition 3 - Precise Definition of One Sided Limits

1. Assume the domain of f contains an interval (c,d) to the right of c. We say that $f(x)$ has a right-handed limit L at c, and write

$$\lim_{x\to c^+} f(x) = L$$

If for every number $e > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < e \text{ whenever } c < x < c + \delta$$

2. Assume the domain of f contains an interval (b,c) to the left of c. We say that f has left-handed limit L at c, and write

$$\lim_{x\to c^-} f(x) = L$$

if for every number $e > 0$ there exists a corresponding number $\delta > 0$ such that

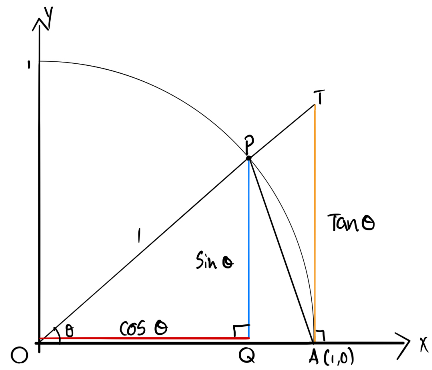
$$|f(x) - L| < e \text{ whenever } c - \delta < x < c$$

1.11 Theorem 6 - Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$

The main thing we need to know about this theorem is that it is derived from **The Squeeze Theorem**. As explained in the Squeeze Theorem we are going to find an upper and a lower function which will allow us to squeeze the middle function as $\theta \to 0$. We will first show the theorem followed by the proof.

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \text{ (θ is in radians)}$$

Proof



The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\frac{\pi}{2}$. Notice that

$$\text{Area } \Delta\text{OAP} < \text{area of sector OAP} < \text{area } \Delta\text{OAT}$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta\text{OAP} = \frac{1}{2}\text{base} \cdot \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector OAP} = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2}$$

$$\text{Area } \Delta\text{OAT} = \frac{1}{2}\text{base} \cdot \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive, since $0 < \theta < \frac{\pi}{2}$:

$$0 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocal reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+} \cos\theta = 1$, the Squeeze Theorem gives

$$\text{Since the } \lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin\theta$ and θ are both *odd functions*. Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y axis. This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-} \tfrac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+} \tfrac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem number 5.

1.12 Definition 4 - Continuity

In simple example continuity would be drawing a line without lifting up the pencil at any point between an interval. Unfortunately however, continuity can get more complex. In order to understand continuity better lets first look at the definition and then practice with a figure.

Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

The function f is **continuous** at c if

$$\lim_{x \to c} f(x) = f(c).$$

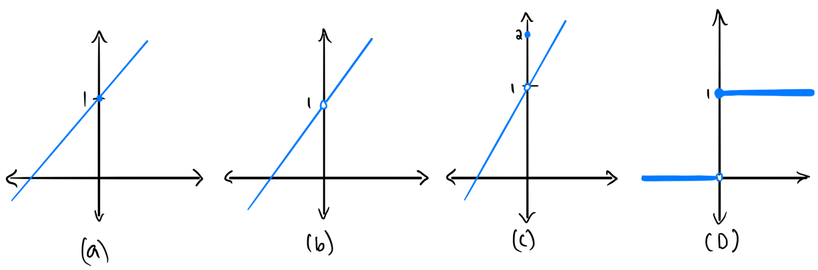
The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** i

$$\lim_{x \to c^-} f(x) = f(c).$$

Given the following figure lets determine which graphs are continuous as $x = 0$ and which type of continuity do they have



1. Graph A is the perfect simple example of continuity as $x = 0$. As stated above it looks like a line drawn with a pencil that was never lifted up.
2. Graph B is not continuous at $x = 0$. Notice the hole in the graph as $x = 0$.
3. Graph C be is not continuous at $x = 0$ either. Notice how graph c has a hole at $x = 0$ and instead of giving the value 1, we jump to the value 2.
4. Graph D is the case of a jump discontinuity. Notice that the one-sided limit exist but they both have different values.

1.13 Theorem 7 - Properties of Continuous Functions

This theorem if looked at carefully, exhibits the same algebraic properties as that of limit laws. Could they possibly be connected? Think on what a it means to be a continuous function. Do you believe a non-continuous functions can have this properties?

This theorem states that if f and g are continuous at $x = c$, then the following algebraic combinations are continuous at x=c.

1. Sums: f+g
2. Differences f - g
3. Constant Multiples: $(k \cdot f)$ for any number k
4. Products: $f \times g$
5. Quotients: $\frac{f}{g}$, provided $g(c) \neq 0$
6. Powers: f^n, n a positive integer
7. Roots: $\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer

1.14 simplifying

Up until now,

1. We saw the Epsilon Delta definition and explained how it is similar to rise over run. A more formal definition would state that $\pm\delta$ is the error from the point c and that $\pm e$ is the error from the $f(x)$ output y.
2. We began to look at limits from both the right and the left. Here is a brief summery.

   If the limit as x approaches a value c from the left and right are different, then the limit does not exist.

   If the limit as x approaches a value c from the left and right are similar, then the limit exists.

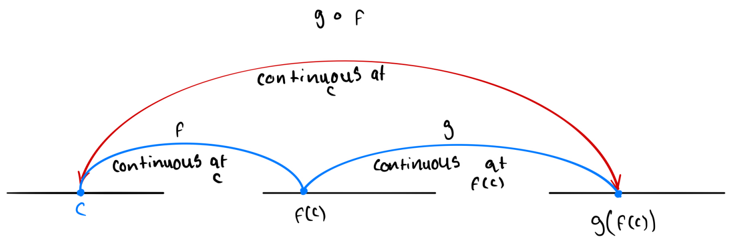
   If f is not defined at the right or the left and has a limit domain endpoint (this was the case when we evaluated for $x = 0$ and 4) Then the limit will be equal to the side in which it exists.

3. We used the Squeeze Theorem to show that the Limit of the ratio $\sin\theta/\theta$ as $\theta \to 0$
4. We introduced the idea of continuous functions and how they like limits are analyzed from the right and left. A basic method to understand continuity is by drawing a line without lifting up the pencil. Otherwise, the function may or may not be continuous.

1.15 Theorem 8 - Compositions of Continuous Functions

This Theorem states that if f is continuous at c and g is continuous at at $f(c)$, then the composition $g \circ f$ is continuous at c.

In order to view this through a different lens lets analyze the following graph and discuss why this theorem is intuitive.



Due to the fact that x is close to c, then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$. The continuity of compositions holds for any finite number of compositions of functions. The only requirement is that each function be continuous where it is applied.

1.16 Theorem 9 - Limits of Continuous Functions

Usually when we think of a function $f(x)$ a common thought is that if we substitute a value for x we will get an output y. It also happens to be that this works backwards if f(x) is continuous at that point. Basically If we have an output and know the function $f(x)$ then we can find the value of x.

The proper definition is the following:If $\lim_{x \to c} f(x) = b$ and g is continuous at the point b, then

$$\lim_{x \to c} g(f(x)) = g(b).$$

Proof Let $e > 0$ be given. Since g is continous at b, there exists a number $\delta_1 > 0$ such that

$$|g(y) \text{ - } g(b)|\ e \text{ whenever } 0 < |y \text{ - } b| < \delta_1$$

Since $\lim_{x \to c} f(x) = b$, there exists a $\delta > 0$ such that

$$|f(x) \text{ - } b| < \delta_1 \text{ whenever } 0 < |x \text{ - } c| < \delta.$$

If we let $y = f(x)$, we then have that

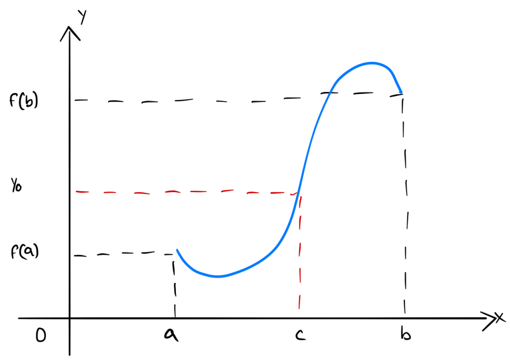
$$|\text{y - } b| < \delta_1 \text{ whenever } 0 < |x \text{ - } c| < \delta,$$

Which implies from the first statement that $|g(y) \text{ - } g(b)| = |g(f(x)) \text{ - } g(b)| < e$ whenever $0 < |x \text{ - } c| < \delta$. From the definition of limit, it follows that $\lim_{x \to c} g(f(x)) = g(b)$. This gives the proof for the case where c is an interior point of the domain f. The case where c is an endpoint of the domain is entirely similar, using an appropriate one-sided limit in place of a two-sided limit. This completes the proof.

1.17 Theorem 10 - The Intermediate Value Theorem for Continuous Functions

Probably the simplest theorem, the Intermediate Value Theorem states that

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(c)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Basically this theorem tells us that given a continuous interval lets assume from [1 to 2] there will be a value c that is in between. This is very intuitive as we have infinite many values between [1 to 2]. For instance we have 1.1 or 1.01 or 1.001 and so on.

1.18 Simplifying

Up until now we have talked about continuous functions and how the composition between two of them is continuous. We have also talked about working backwards given a y value to find the original x value. Lastly we have talked about how in the interval of a continuous function there is a value c, or better said there are infinity many of them. Now however, we will **shift our perspective** onto the last part of limits we need to know. Taking the limit as $x \to \infty$. It is very common for us to substitute a finite value into a function $f(x)$, nonetheless, we can do more than substitute finite values. We can substitute infinite values (∞). When we do so we will now look out for **horizontal and vertical asymptotes** because they will serve as our limits for the function. Recall **Theorem 3**.

1.19 Definition 5

Here we have a very detailed definition on taking the limit as $x \to \infty$ from both the right and left hand side.

(A) We say that $f(x)$ has the limit L as x approaches infinity and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number e >0, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - l| < e \text{ whenever } x > \text{M}.$$

(B) We say that $f(x)$ has the limit L as x approaches negative infinity and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number e >0, there exists a corresponding number N such that for all x in the domain of f

$$|\ f(x) - L\ | < e \text{ whenever } x < \text{N}.$$

1.20 Theorem 11

All the Limit Laws in Theorem 1 are true when we replace $\lim\limits_{x \to c}$ by $\lim\limits_{x \to \infty}$ or $\lim\limits_{x \to^- \infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

1.21 Definition 6 - Horizontal Asymptote

A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to\infty^-} f(x) = b.$$

Example given $f(x) = \dfrac{5x^2 + 8x - 3}{3x^2}$ can you find the limit as $x \to \infty$?

$$\text{Step 1: } \frac{5x^2}{3x^2} + \frac{8x}{3x^2} - \frac{3}{3x^2}$$

$$\text{Step 2: } \lim_{x\to\infty} (\frac{5x^2}{3x^2} + \frac{8x}{3x^2} - \frac{3}{3x^2})$$

$$\text{Step 3: } \lim_{x\to\infty} \frac{5}{3} + \frac{8}{3x} - \frac{3}{3x^2}$$

$$\text{Step 4: } \frac{5}{3} + 0 + 0$$

This means that as x approaches ∞ the graph will go to 5/3. This is our Horizontal Asymptote(H.A) The short cut to know whether we have a H.A is the following. Given $\dfrac{af(x)^M}{bf(x)^N}$ if M is greater than N, then there is no H.A, if M is equal to N divide the coefficients a/b, if N is greater than M, then the H.A is at y = 0.

1.22 Definition 7 - Precise Definition Of Infinite Limits

(A) We say that $f(x)$ approaches infinity as $f(x)$ approaches c, and write

$$\lim_{x\to c} f(x) = \infty$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that

$$f(x) > \text{ whenever } 0 < | x - c | < \delta.$$

(B) We say that $f(x)$ approaches negative infinity as x approaches c, and write

$$\lim_{x\to c} f(x) = \infty,$$

if for every negative real number -B there exists a corresponding $\delta > 0$ such that

$$f(x) < \text{-B whenever } 0 < | x - c | < \delta$$

1.23 Definition 8 - Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty.$$

Example Given $y = \dfrac{x+3}{x+2}$ find the Horizontal and vertical Asymptote (V.A).

In this case our Horizontal (Asymptote will be equal to $y = 1$ as the degrees of x are the same and their coefficient is 1. Thus $1/1 = 1$. The Vertical Asymptote however, is even easier to calculate in this scenario. We will always equal the denominator of a rational function to zero. Thus, we make the claim that $x + 2 = 0$. Then we isolate for x and get that $x = 2$ for our Vertical Asymptote.